AN EXTENSION OF FEUERBACH'S THEOREM

By F. Morley

JOHNS HOPKINS UNIVERSITY

Received by the Academy, February 17, 1916

Feuerbach's theorem, that the four circles which touch three lines also touch a circle, may be stated thus: given four orthocentric points, forming four triangles, the 16 in-circles of these triangles touch the circle F on the diagonal points.

Now each in-circle and the omitted one of the four points is a degenerate curve of class three on the absolute points $I\ J$. There is further a rational curve of class three, on the six joins of the four points and touching the line infinity at $I\ J$, which touches F three times. Thus the theorem is suggested: *All circular line-cubics on the joins of four orthocentric points touch the Feuerbach circle.*

A proof is as follows. It is convenient to state the algebra dually. That is, we have 4 lines 1, $\pm$ 1, $\pm$ 1 and a pair of lines, ζ and $1/\zeta$, apolar to all conics on the 4 lines. Two point-cubics on the six joins of the 4 lines meet again at 3 points $x\ y\ z$, which are points of contact of a tritangent conic of either cubic. When x and y are given, z is rationally known; and when x is given and z moves on a line ζ, we know from the theory of the Geiser transformation that y moves on a rational quartic $\rho_x{}^4$ which has a triple point at x. There is then a connex of the form

$$\zeta\ x^4\ y^4$$

where z is on ζ. And if ξ be the join of x and y this connex is of the form

$$\zeta\ \xi^3\ x\ y. \tag{1}$$

If ζ be 1, 0, 0, the quartic in y is the two lines

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ 0 & 1 & \pm 1 \end{vmatrix}$$

and the conic on x and the 4 other points, that is

$$(\xi_1{}^2 - \xi_2{}^2) \begin{vmatrix} x_0{}^2 - x_1{}^2 - x_2{}^2, x_1 x_2 \\ y_0{}^2 - y_1{}^2 - y_2{}^2, y_1 y_2 \end{vmatrix} .$$

Hence the connex is explicitly

$$\begin{aligned} \sum^{3} \zeta_0\, (\xi_1{}^2 - \xi_2{}^2)\ \{\ (x_0y_1 + x_1y_0)\ \xi_1 - (x_0y_2 + x_2y_0)\ \xi_2 \\ + 2\, (x_1y_1 - x_2y_2)\ \xi_0\ \} = 0. \end{aligned} \tag{2}$$

We now find where this curve in y meets the line $1/\zeta$. That is, we eliminate y from

$$\Sigma = 0, (\xi y) = 0, (y/\zeta) = 0.$$

We have then

$$\Sigma\zeta_0 (\xi_1^2 - \xi_2^2) \begin{vmatrix} x_1\xi_1 - x_2\xi_2, & x_0\xi_1 + 2x_1\xi_0, & -x_0\xi_2 - 2x_2\xi_0 \\ \xi_0 & \xi_1 & \xi_1 \\ 1/\zeta_0 & 1/\zeta_1 & 1/\zeta_2 \end{vmatrix} = 0,$$

or since $(x\xi) = 0$

$$\Sigma (\zeta_1/\zeta_2 - \zeta_2/\zeta_1) x_0 (\xi_0^2 - \xi_1^2) (\xi_0^2 - \xi_2^2) = 0$$

or if a be the join of ζ and $1/\zeta$,

$$\Sigma a_0 x_0 (\xi_0^2 - \xi_1^2) (\xi_0^2 - \xi_2^2) = 0. \tag{3}$$

The values of ξ common to this equation and $(x\xi) = 0$ give the intersections y of ρ_x^4 and $1/\zeta$. Thus when $(x\xi)$ is a line of (3) then as z moves on ζ the curve ρ_x^4 touches $1/\zeta$ at the point y, and x is on the envelope sought.

Now the quartic (3) is two conics on the lines $1, \pm 1, \pm 1$. And when

$$\Sigma \sqrt{a_i x_i} = 0, \tag{4}$$

the two conics become one conic R whose equation is

$$\Sigma \sqrt{a_i x_i}\, \xi_i^2. \tag{5}$$

The conic (4) occurs then twice in the envelope, the other factors being

$$[x_0^4 + 2x_1^2x_2^2]^2,$$
$$x_0x_1x_2,$$

and the cubic of the system with a double point at a, namely

$$\Sigma\, x_0\eta_1\eta_2$$

where η is the join of x and a.

The conic (4) is the Feuerbach conic F, for it is on the diagonal lines of the four lines, and having the line equation

$$\Sigma\, a/\xi = 0,$$

it is on ζ and $1/\zeta$.

The construction of the cubics which touch both ζ and $1/\zeta$ is then as follows. Take a point x on F, and draw from x the two tangents to the conic R. The diagonals of this line-pair and the pair ζ and $1/\zeta$ give the points of contact of the two cubics. If the diagonals meet at b,

then x and b are apolar to ζ and $1/\zeta$; and the line bx, being the polar of a as to R, has the equation

$$\Sigma\, ay/\sqrt{ax} = 0,$$

and is the tangent of F at x.

Dually then, ζ and $1/\zeta$ being the absolute points, the conic F the Feuerbach circle, and the conic R a rectangular hyperbola on the four given orthocentric points, and having its centre c on F, if the common diameter of F and R meet R at points dd', then these points are double foci of circular curves of class 3 on the 6 lines; the circles with centres d and d' and touching F at c are the tritangent conics; and the two cubics touch F at c.

DEFORMATIONS OF TRANSFORMATIONS OF RIBAUCOUR

By L. P. Eisenhart

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

Received by the Academy, February 5, 1916

When a system of spheres involves two parameters, their envelope consists in general of two sheets, say Σ and Σ_1, and the centers of the spheres lie upon a surface S. A correspondence between Σ and Σ_1 is established by making correspond the points of contact of the same sphere. In general the lines of curvature on Σ and Σ_1 do not correspond. When they do, we say that Σ_1 is in the relation of a transformation of Ribaucour with Σ, and vice-versa. For the sake of brevity we call it a transformation R.

It is a known property of envelopes of spheres that if the surface of centers S be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before deformation.[1] Ordinarily when S for a transformation R is deformed, the new surfaces Σ' and Σ'_1 are not in the relation of a transformation R. Bianchi[2] has shown that when S is applicable to a surface of revolution, it is possible to choose spheres so that for every deformation of S the two sheets of the envelopes of the spheres shall be in the relation of a transformation R, and this is the only case in which S can be deformed continuously with transformations R preserved. The only other possibility is that in which it is possible to deform the surface of centers of a transformation R in one way so that the sheets of the new envelope shall be in the relation of a transformation R. It is the purpose of this paper to determine this class of transformations R.